Problem 1. Harry challenges Hermione to a battle of brainpower. He tells her to compute

$$1 * 1 + 1 * 2 + 2 * 2 + 2 * 3 + 3 * 3 + 3 * 4 + 4 * 4 + 4 * 5$$

What should Hermione answer?

Solution. $1 + 2 + 4 + 6 + 9 + 12 + 16 + 20 = 70$ □

Problem 2. Ron is very hungry and eats 210 chocolate frogs in 3 days! There are 144 calories per frog. How many calories is that per minute?

Solution. 210 chocolate frogs in 3 days = 210 chocolate frogs in 72 hours = 210 chocolate frogs in 4320 minutes = 30240 calories in 4320 minutes. $\frac{30240}{4320} = 7$ □

Problem 3. The letters a, b, c, d represent 4 different whole numbers. If $a = 8$, $a * b = a$, $b + c = a$, and $c * (d - 1) = a * c$, then what is the value of d?

Solution. We will look at the equation:

$$\begin{aligned} c * (d - 1) &= a * c \\ d - 1 &= a \\ d - 1 &= 8 \\ d &= 9 \end{aligned}$$

□

Problem 4. The Sorting Hat has started to give riddles to sort students into their houses! It poses the following question: I can divide 720 without remainder, while I can be divided by the first 6 natural numbers. The sum of my digits is 3, as is the amount of digits. What number am I?

Solution. First, we will begin by finding out the smallest number that can be dividing by the first 6 natural number (1, 2, 3, 4, 5, 6). We find the least common multiple of the first 6 natural numbers which happens to be $60 = 4 * 5 * 3$. However, 60 does not work as it is not a 3 digit number so we then begin increasing by factors of 60. $60 + 60 = 120$. $1 + 2 + 0 = 3$ and 120 is a 3 digit number. Thus, 120 is the answer. □

Problem 5. 5 Gryffindors stand in a circle. Each sends out a number with their wand from 1-6 uniformly at random. Each sends out a number independent of each other. The probability that the sum of the 5 numbers is 7 is $\frac{m}{n}$ where m and n are relatively prime integers. Find $m + n$.

Solution. A sum of 7 can be achieved in 2 different ways from these 5 wands ($1 + 1 + 1 + 1 + 3$ or $1 + 1 + 1 + 2 + 2$). 1, 1, 1, 1, 3 can be created in $\binom{5}{1} = 5$ ways as we need to select which wand sends out 3 and all of the others will send out 1. Similarly, 1, 1, 1, 2, 2 can be creating in $\binom{5}{2} = 10$ ways as we need to select the 2 wands that will send out a 2 and the other 3 will send out a 1. This gives a total of 15 ways. The total number of different possibilities is $6^5 = 7776$. Thus, the probability is $\frac{15}{7776} = \frac{5}{2592}$. Finally, $m + n = 5 + 2592 = 2597$. □

Problem 6. The Hungarian Horntail has been recaptured after the first round! It is restricted to flying in the interior or on the cube. If the Hungarian Horntail can only fly from one vertex to another along a diagonal (including face diagonals), how many different paths (movement from 1 vertex to another regardless of direction) can the Horntail fly on?

Solution. All this question is asking is how many diagonals are there on a cube. Each face has 2 and their are 4 diagonals that travel through the interior of the cube. Thus the final answer is $2 * 6 + 4 = 16$. □

Problem 7. While he's in the Room of Requirement, Harry finds the design for the Sorting Hat, which is a cone! Harry wants to follow the design and starts of with a circle of radius 8. He then cuts out a quarter of the circle and connects the 2 edges of the $\frac{3}{4}$ circle that remains to form a cone. The outer surface area of the Sorting Hat (which does not have a base) is $a\pi$, where a is a whole number. What is a?

Solution. This question sounds complicated but in reality it is very simple. The cone is being constructed from the three quarters circle that remains. Thus, the surface area of the cone is just the area of the remaining circle which is just $\frac{3}{4} * 8^2\pi = 48\pi$. □

Problem 8. The Ministry of Magic's cars are once again being used to transport Harry and the Weasleys to King's Cross! However, the magic they have is dying, and so the tires are slowly wearing out. The tires lose 1 mm of diameter for every 50 km traveled, and the original radius of each of the tires is 300 mm. These cars are traveling at a constant speed of 100 km/h. How much time would the trip to King's Cross need to take for the tires to be at a radius of 250 mm?

Solution. In order for a tire to lose 50 mm of radium, they need to lose 100 mm of diameter. Thus, the car would need to travel $50 * 100 = 5000$km. Thus, the amount of time the trip takes is $5000/100 = 50$. □

Problem 9. Professor Vector is fed up with his students. They refuse to do their work or study for the exams, except for Hermione. So, Hermione gets a chance at extra credit! Professor Vector asks the following question: what is the remainder when 3^{13} is divided by 4?

Solution. For small powers of 3, we see that the sequence $3, 9, 27, 81, 243, \ldots$ has remainders of $3, 1, 3, 1, 3, \ldots$ when divided by 4. The remainder alternates between 3 and 1 for odd and even powers, respectively, so the remainder of 3^{13} when divided by 4 is 3.
A more formal solution is that $3^{13} \equiv (-1)^{13} \mod 4 \equiv -1 \equiv 3 \mod 4$ using modular arithmetic. □

Problem 10. In Hagrid's Care of Magical Creatures class, he gives 5 students a Niffler, a cute creature that can seek out treasure. Hagrid has hidden 3 indistinguishable coins within his magical bag that is enchanted by the Extension Charm, making it have incredible space inside. In how many ways can these 3 coins be retrieved by the 5 Nifflers?

Solution. This is an application of the Stars-and-Bars method. As a Niffler can get 0 coins, the number of ways to count this is $\binom{5+3-1}{3} = \binom{7}{3} = 35$. □

Problem 11. In Harry's first year, Professor Vector wanted to guard the Sorcerer's Stone using an arithmancy question. For some reason, it was never used. However, it's always a question on the final exam. Here is the question: Professor Vector has his hands on triangles. Each of his triangles has a perimeter of 9, has integer side lengths, and are not congruent to any other triangle. What is the maximum number of triangles he could have?

Solution. We use the Triangle Inequality Theorem here, which states that $c < a + b$ where c is the longest side length of the triangle and a, b are the other side lengths. As $a + b + c = 9$, the largest side length can be at most 4, as for lengths ≥ 5 the Triangle Inequality Theorem will not be satisfied. Checking through various triples, we see that the only unique triples that work for (a, b, c) are $(3, 3, 3), (4, 4, 1)$, and $(4, 3, 2)$, for a total of 3 triangles. $\square$

Problem 12. Harry is playing against Ravenclaw! While looking for the Snitch, Harry goes against the wind, which is at an unknown speed "x" MPH. It takes him 15 seconds to travel $\frac{1}{4}$ of a mile. Harry swerves and goes in the opposite direction as he tries to avoid a bludger, and now he goes in the same direction as the wind! Here, it takes him 8 seconds to travel $\frac{1}{4}$ of a mile. In the absence of wind, Harry flies at $\frac{m}{n}$ miles per minute, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Let the speed in miles per minute that Harry flies at be denoted by S, and the speed of the wind in miles per minute be W. When going against the wind, we have $S - W = \frac{1}{4} * \frac{60}{15} = 1$, and when going with the wind, we have $S + W = \frac{1}{4} * \frac{60}{8} = \frac{15}{8}$. Adding both equations together, we have $2S = 1 + \frac{15}{8} = \frac{23}{8}$, so $S = \frac{23}{16}$, and $m + n = 23 + 16 = 39$. $\square$

Problem 13. Fred and George have accidentally mixed an infinite deck of exploding cards and an infinite deck normal cards! They decide to play a game of luck with this mixed deck. The rules are as follows:

1. The first person to have 3 cards explode in their hand loses.
2. On each round, there is a $\frac{1}{3}$ probability that Fred draws one card from the deck, there is a $\frac{1}{3}$ probability that George draws a card from the deck, and a $\frac{1}{3}$ probability that neither of them draw a card and the round goes by with nothing happening.

The probability that the game lasts at most 3 rounds is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. There are only two possibilities that the game lasts at most 3 rounds: either Fred draws 3 exploding cards in a row, or George draws 3 exploding cards in a row. Both have an equal probability of doing so. In order for Fred to get 3 exploding cards, he has a $\frac{1}{3}$ chance of drawing a card, and a $\frac{1}{2}$ chance of getting an explosive card instead of a normal card. Thus, Fred gets 3 exploding cards with probability $(\frac{1}{6})^3 = \frac{1}{216}$, and George has the same probability. The total probability is $2 * \frac{1}{216} = \frac{1}{108}$, and $m + n = 1 + 108 = 109$ $\square$

Problem 14. Someone is trying to get into Dumbledore's office, which has a history of having extremely obscure passwords. When looking over Dumbledore's shoulder one day, the intruder only noticed the first and last digit of the 4 digit password. The first digit was 7 and the last digit was 4. He also knows that Dumbledore loves arithmancy, and would make his code divisible by 2,3,4,6,8, and 9. What is the sum of all possible passwords that Dumbledore could have?

Solution. Dumbledore's password is of the form $7XY4$, where X and Y represent digits from 0 to 9. We first look at the divisibility by 9 rule, which gives us that $7 + X + Y + 4 = 11 + X + Y$ is divisible by 9. The two possible cases are then $X + Y = 7$ or $X + Y = 16$ as $0 \leq X + Y \leq 18$. We also take a look at the divisibility by 4 rule, which gives us that $10Y + 4$ has to be divisible by 4. We see that Y must be even for this to be true. We proceed with the following two cases:
Case 1: $X + Y = 7$. The only pairs of (X, Y) that may work are $(7, 0), (5, 2), (3, 4), (1, 6)$. We now check for which ones satisfy divisibility by 8. 7704 and 7344 are the only two that work.
Case 2: $X + Y = 16$. The only pair of (X, Y) that may work is $(8, 8)$. As 7884 is not divisible by 8, no solutions exist for this case.
Thus, the total sum is $7704 + 7344 = 15048$. $\square$

Problem 15. Harry has just Apparated from 1 location on the coordinate plane to another! He Apparated from $(5,2)$ to $(-2,1)$, which also happens to be a 90 degree counterclockwise rotation about a 3rd coordinate, (a,b), on the coordinate plane. Compute a^2+b^2.

Solution. We translate the coordinate $(5,2)$ by $(-a,-b)$ so that we can do a CCW rotation around the origin. This is $(5-a, 2-b)$, and after performing a 90° CCW rotation about the origin, this turns into $(b-2, 5-a)$. Translating back, our final point is $(a+b-2, -a+b+5)$, which corresponds to $(-2,1)$. Solving this two-equation system, we find that $(a,b) = (2,-2)$, and $a^2+b^2 = 8$ □

Problem 16. Harry is fighting a Hungarian Horntail, and his attacks have a 50% chance of hitting the dragon in its weak spot, otherwise it hits the dragon's scaly flesh. When Harry hits the dragon's weak spot, it takes away $\frac{1}{5}$ of its health, and when he hits the dragon's flesh, it takes away $\frac{1}{10}$ of its health. When the dragon attacks, it takes away $\frac{1}{7}$ of Harry's health. The order of the attacks is always: Harry, Dragon, Harry, Dragon... The probability that the dragon dies before Harry is $\frac{a}{b}$ where a and b are relatively prime positive integers. Find $a+b$. (Note: Harry does not attack the dragon once it is dead.)

Solution. Harry must finish killing the Hungarian Horntail in 7 moves or less, as otherwise he will be killed first. We use complementary counting; we find the probability that Harry dies first, and then find the complement of that probability to solve the question. Harry will die with the following combination of attacks: 7 $\frac{1}{10}$ attacks, 6 $\frac{1}{10}$ + 1 $\frac{1}{5}$ attack, or 5 $\frac{1}{10}$ attacks + 2 $\frac{1}{5}$ attacks. The first case has a $(\frac{1}{2})^7 = \frac{1}{128}$ chance of happening, the second case has a $(\frac{1}{2})^7 * \binom{7}{1} = \frac{7}{128}$ of happening, and the third case has a $(\frac{1}{2})^7 * \binom{7}{2} = \frac{21}{128}$ chance of happening. The total probability of Harry dying first is $\frac{1+7+21}{128} = 29128$. Thus, the probability of Harry killing the dragon is $1 - \frac{29}{128} = \frac{99}{128}$, so $a+b = 99+128 = 227$. □

Problem 17. One of the tasks of the Triwizard Tournament is to find the 2021st term of a certain sequence. Something special about this sequence is that for every positive integer n, the average of the first n terms of the sequence is n. What is the correct answer to this task?

Solution. Let the sequence be $a_1, a_2, a_3, \cdots, a_n$. First let $n=1$. The average of the first 1 terms is 1, so $a_1 = 1$. Next let $n=2$. The average of the first 2 terms is 2, so $\frac{a_1+a_2}{2} = 2$. This is $1+a_2 = 4$, or $a_2 = 3$. For $n=3$, we have $\frac{a_1+a_2+a_3}{3} = 3$, and we see that $a_3 = 5$. It can be shown that a_n is just the nth odd number, or that $a_n = 2n-1$. Plugging in $n = 2021$, we have $a_{2021} = 2*2021-1 = 4041$.

□

Problem 18. Seamus is playing with a 2-by-2 Rubik's cube which is made up of 8 1×1 cubes. Each of the 6 faces of the Rubik's cube are a distinct color. However, one day a Blasting Curse breaks the cube apart into the 8 smaller 1×1 cubes. Seamus wants to put the cube back together placing each of the pieces one by one. Due to how the cube is constructed, he is only able to place a piece if an adjacent piece is already placed (with an exception of the first piece). How many different ways can he choose to put this cube back together so that it is identical to how it was before it was destroyed?

Solution. We have 8 choices for the first cube that Seamus picks up. Then, he has 3 choices for the second cube and 4 for the third cube, yielding us 96 ways to first construct an L made up of three cubes. Now, note that there are 4 places to put the fourth cube. If Seamus decides to not place the cube on top of the center of the L, then we can observe that each of the 4 remaining spots can be filled in any order. Otherwise, if Seamus places the cube on the top of the center of the L, he has 3 places for the next cube, and then he can fix the cube with the last 3 pieces in any order. This yields $96*(3*24+18) = 96*90 = 8640$ ways that Seamus can fix the cube.

□

Problem 19. Harry and Draco have decided to have another Wizard's Duel, but this time, it's an academic duel! They decide to play a game. They start with three concentric circles of radii 1, 2, and 3 centered at the origin of the complex plane. Harry will first place point A on the circle of radius 1, then Draco will place point B on the circle of radius 2, and finally Harry will place point C on the circle of radius 3. Harry wants to maximize the area of $\triangle ABC$, while Draco wants to minimize the area. If both Harry and Draco play optimally, the area of $\triangle ABC$ is $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $m + n$.

Solution. It does not matter where Harry places the first point due to symmetry, so let $A = (1, 0)$. Because Draco wants to minimize the area of the triangle, he will pick the point closest to A because this will minimize the side length (and thus the area) of the triangle. This is evidently $B = (2, 0)$. Now Harry must pick a point on the circle of radius 3. As AB lies on the x-axis, consider only the y-component of $C = (x, y)$. Regardless of the value of x, the area of the triangle will still be $\frac{1}{2}(1)(y)$, and so Harry will choose a point to maximize y. This is $(0, 3)$, so the area of $\triangle ABC = \frac{3}{2}$ and $m + n = 3 + 2 = 5$.

□

Problem 20. Harry is eating a slice of cheese which is a triangle with side lengths 6, 8, and 10. He is in arithmancy class and has just learned about the properties of triangles. He is trying to find the distance between the center of the inscribed circle and the circumscribed circle of the slice of cheese. The answer to this question can be written in the format $\sqrt{a}$, find a?

Solution. Construct a triangle ABC such that $A = (0, 0), B = (6, 0)$, and $C = (0, 8)$. This is a right triangle with side lengths $6, 8$ and 10. A property of the circumcenter of a triangle is that it lies on the hypotenuse and divides it into two equal parts. Thus, we know that O (which will denote the circumcenter) is the midpoint of hypotenuse BC, or $(3, 4)$. To find the incenter I, we can use formulas for area of a triangle. The area of a triangle can be written as rs, where r is the inradius and s is the semiperimeter, so $\frac{1}{2}(6)(8) = r(\frac{6+8+10}{2})$ leads to $r = 2$. Because the incircle is perpendicular to all the sides of the triangle, a radius drawn from I to AB and AC shows that $I = (2, 2)$. By the distance formula, $OI = \sqrt{(3-2)^2 + (4-2)^2} = \sqrt{5}$. Thus, $a = 5$. □