Problem 1. Harry needs your help evaluating this expression:

$$111 * 10 + 11 + 1$$

Can you help him?

Solution.

$$(111 * 10) + 10 + 1 = 1110 + 11 + 1 = \mathbf{1122}$$

□

Problem 2. Circle 1 is inscribed inside of square 1, and square 2 is inscribed in circle 1. What is the ratio between the areas of square 1 and square 2?

Solution. We begin by finding the ratio between the area of circle 1 to square 1. We know the diameter of circle 1 is equal to the height of square 1 and we can set this value to x. To find the area of square 1, we multiply the square's height by its width and get x^2. We also know that the diagonal of square 1 is equal to the diameter of circle 1 which we called x. To find the side length of square 2, we can use the Trigonometry property of a 45-45-90 triangle, so we know the side length is $\frac{x}{\sqrt{2}}$. We find the area of this square to be $\frac{x^2}{2}$. Therefore, the ratio of square 1 to square 2 is $\frac{x^2}{x^2/2}$ which can be simplified to **2**. □

Problem 3. Professor Trelawney is teaching the Gryffindors about planetary alignments, and says that if two planets can be connected by a radius drawn from the Sun, they are in conjunction. Conjunctions can happen between 2 or more planets, but here, we consider conjunctions of only 2 planets. Given that there are 8 planets, how many different conjunctions can there be? (Location of the conjunction does not matter)

Solution. We know that any two points around a circle can be connected by a straight line, so the number of conjunctions connecting two planets is equivalent to $C(8, 2)$ which is the total number of combinations. Using the formula for combinations, we find the total number of conjunctions to be $\frac{8!}{2!(8-2)!}$ which is equal to **28**.

□

Problem 4. Luna and Hermione have been punished for talking during Arithmancy class. They are being required to cut the grass on the Quidditch field with scissors. By herself Luna would take 5 days to finish cutting the entire field while it would take Hermione 4 days to cut the grass. How many hours does it take for them to cut it when they work together (round to the nearest whole number)?

Solution. We know that Luna takes 5 days to cut all the grass, and Hermione will take 4 days to cut all the grass. Thus, we know that Luna can cut $\frac{1}{5}$ of the grass in one day and Hermione can cut $\frac{1}{4}$ of the grass in one day. Together, they can cut $\frac{1}{5}+\frac{1}{4}= \frac{9}{20}$ of the grass in one day. To cut all the grass, we take the reciprocal of $\frac{9}{20}$ which tells us that if Luna and Hermione work together, they will take $\frac{20}{9}$ days. To find the number of hours, you multiply that value by 24 which is equal to $\frac{160}{3}$ which, if rounded to the nearest whole number, is **53**. □

Problem 5. There was a large debate concerning the order of the obstacles guarding the Sorcerer's Stone in Harry's first year. There were seven different obstacles. There were two rules, however:

1. Hagrid's obstacle (one of the obstacles) had to be first.

2. Snape's obstacle (another one of the obstacles) had to be last.

How many possible orders were there for the obstacles?

Solution. We can disregard obstacle 1 and obstacle 7, because they only have one possible case, and therefore, do not produce a difference in possible orders. We can now find the total possible number of obstacles by finding 5! which is equal to **120**.

□

Problem 6. A composite number "n" is known to be magical if the number of its divisors is divisible by 7. What is the least magical number?

Solution. Case 1 (7 factors) : For a number to have 7 positive factors, we need to find the positive factors of 7, and we know those are 7 and 1 and subtract each of them by 1. This is equal to 6 and 0. We now set these to be the powers of distinct prime numbers, a and b. To find the smallest number with 7 factors, we would set the value of a to be the smallest prime number which is 2 and b to the power of any value is just 1. Thus, the smallest number with exactly 7 factors is **64**.

We know that all numbers smaller than 64 have less than 14 factors because for a number to have 14 factors it is equal to $a^6 * b^1$ with a and b being prime numbers using the property from Case 1. To find the smallest value that has 14 factors, we can plug 2 and 3 into the values of a and b and this is 102. As the number of factors increase, the integer increases in value, and thus, the smallest number with factors divisible by 7 is equivalent to **64**.

□

Problem 7. Gilbert the Gecko lives on (1, 3) on the coordinate plane. His house is part of a square neighborhood with vertices (0,0), (4, 0), (4, 4), and (0, 4). Everyday, he travels either up, down, left, or right randomly with equal probability by 1 unit. The probability that, after at most 3 moves, Gilbert leaves his neighborhood (goes outside the square) is $\frac{m}{n}$ where m and n are relatively prime integers. Find $m+n$. Note: Once he leaves the neighborhood he does not move anymore. Being on the border is also leaving the neighborhood.

Solution. We know that at the point (1,3), Gilbert can either go up or left and exit his neighborhood. This would provide us with a $\frac{1}{2}$ probability that Gilbert exits the neighborhood in one step. For the other 2 cases at P(2,3) and P(1,2), Gilbert has one possible way for him to exit the square which provides us with a $\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ probability that Gilbert exits the neighborhood in 2 steps. If Gilbert ends up at at a point inside the neighborhood after his second step, there is a $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ chance that he exits the neighborhood in his third step. If we add up the total probabilities, we get 3 as m and 4 as n which gives us a sum of **7**.

□

Problem 8. Three shaded circles of radius 1 are centered on the vertices of an equilateral triangle of side length 10. The area of the triangle outside the circles (the area of the non shaded region of the triangle) can be written as $a\sqrt{b} - \frac{\pi}{c}$, where a and c are positive integers and b is a square-free positive integer. Find the value of $a+b+c$.

Solution. We know that the area of each individual circle is equal to π. The sectors that overlap the triangle are 60 degrees which is $\frac{1}{6}$ of the area of each circle. Since there are three circles, the total overlapping area is $\frac{\pi}{2}$. The area of an equilateral triangle with side length 10 is equal to $\frac{100\sqrt{3}}{4}$ or $25\sqrt{3}$. To find the area inside the triangle but not overlapped by the circles, we get $25\sqrt{3} - \frac{\pi}{2}$. If we add up the values 25, 3, and 2, we get that the answer is **30**.

□

Problem 9. Find the number of real solutions to $7\cos(\pi x) = |x|$.

Solution. We know that the amplitude of $7cos(\pi x)$ is equal to 7 and the period of the function is equal to 2. Before the x-values reaches 2π, $y = x$ and $y = 7cos(\pi x)$ intersect 7 times. Since the right side of the function is equal to the absolute value of x, we know that we have to multiply this value by 2 to account for negative x-values. Therefore, we get **14** real solutions. □

Problem 10. Ron has three galleons in his pocket. Two of the galleons are fair, and one galleon is double-headed. For six iterations, he picks a Galleon at random from his pocket and flips it and proceeds to put it back in his pocket. The probability that he gets at least 3 heads from 6 flips can be expressed as $\frac{m}{n}$. Find m+n.

Solution. We know that the probability of getting a head on one flip is $(\frac{1}{3})(1)$+$(\frac{2}{3})(\frac{1}{2})$ which is equal to $\frac{2}{3}$. The probability of getting 3 heads in one combination is equal to $(\frac{2}{3})(\frac{2}{3})(\frac{2}{3})(\frac{1}{3})(\frac{1}{3})(\frac{1}{3})$= $\frac{8}{729}$. If we account for all combinations where there can be three heads in six flips, we get 20 different combinations. We now multiply 20 by $\frac{8}{729}$ to get $\frac{160}{729}$. To solve for m+n, we get **889**.

□

Problem 11. Solve for x:

$$2^x = 8^{3x} * 64^{2x}$$

Solution. We know that 64 is equal to 2^6 and 8 is equal to 2^3, so we can start by converting all parts of the equation to base 2. The converted equation would be equal to 2^x= 2^{9x}×2^{12x}. This is equal to 2^x=2^{12x}. Since the bases are equal to each other, we can now set x to be equal to $12x$ and the only value for which this is true is **0**.

□

Problem 12. There is a circle centered about point A. Radius is drawn from point A to point D. Chord BC intersects AD at point E. BE = 2, EC = 5. If ED = 1, the radius of the circle can be written as $\frac{m}{n}$, where m and n are relatively prime integers. Find $m + n$.

Solution. We know that when two chords intersect, the product of opposing lengths of both chords is equal. We can set Point P to be the endpoint of the diameter created from Point D. We know $EP \times ED = EB \times EC$. If we set the radius to be r, we get the equation: $(2r - 1)(1) = 5 \times 2$. If we solve for r, we get r=11/2. Thus, m and n would be 11 and 2 respectively and the sum would be equal to **13**

□

Problem 13. Harry needs your help evaluating this expression in binary:

$$111 * 10 + 11 + 1$$

Can you help him find the answer in binary?

Solution. First we need to convert the binary expression to base 10. We can do this by solving for each individual term. 111 in binary is equal to $1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$ which is 7 in base 10. 10 in binary is equal to $1 \times 2^1 + 0 \times 2^0$ which is 2 in base 10. 11 in binary is equal to $1 \times 2^1 + 1 \times 2^0$ which is 3 in base 10. Lastly, 1 in binary is equal to 1 in base 10. If we plug these values back into the original equation, we get $7 \times 2 + 3 + 1$ which is equal to 18.

Now, we want to find this value in binary. To find this, we must find the largest number that can be expressed as 2^x that is less than 18 and this is 16 (2^4). The value of 16 in base 2 (binary) would be 10000. Then, we subtract 16 from 18 and get 2. 2 in binary is equal to 10. Thus, 18 in binary would be equal to **10010**.

Furthermore, we also accepted an answer of 10001100010 as that is what you get if you assume the equation is in base 10, compute it, and then convert it into binary.

□

Problem 14. Regular hexagon 1 is inscribed inside of a circle. Circle 1 is inscribed inside Hexagon 1. Regular Hexagon 2 is inscribed inside of Circle 1. The ratio of the area of Hexagon 1 to Hexagon 2 can be written as $\frac{m}{n}$, where m and n are relatively prime integers. Find $m+n$.

Solution. We draw the diagram. Let the radius of Circle 1 be D. Therefore, half of the diagonal of the hexagon is also D. We notice that a 30-60-90 triangle can be constructed inside of the hexagon, with D being the hypotenuse. Therefore, the radius of the circle inscribed inside of Hexagon 1 is now $\frac{\sqrt{3}*D}{2}$. Therefore, half of the diagonal of Hexagon 2 is now $\frac{\sqrt{3}*D}{2}$. Area is proportional to the length of a side (or diagonal) squared, ratio of the area of Hexagon 2 to Hexagon 1 is $\frac{3}{4}$. Since the question is asking for the ratio of Hexagon 1 to Hexagon 2, it is just the reciprocal, so the fraction is $\frac{4}{3}$. Therefore, the answer is $4+3=7$. □

Problem 15. Professor Trelawney is teaching the Gryffindors about planetary alignments, and says that if two planets can be connected by a line drawn from the Sun, they are in conjunction. Conjunctions can happen between 2 or more planets, but here, we consider conjunctions of any number of planets greater than 1. Given that there are 8 planets, how many different conjunctions can there be? (Location of the conjunction does not matter)

Solution. We notice that this is essentially just $\binom{8}{n}$, where n varies from 2 to 8. We can use complimentary counting. This addition is equal to $2^8 - \binom{8}{1} - \binom{8}{0}$, which is $256 - 9 = \mathbf{247}$. □

Problem 16. Luna, Ron, and Hermione have been punished for talking during Arithmancy class. They are being required to cut the grass on the Quidditch field with scissors. Together, Luna and Ron would take 5 days to finish cutting the entire field, Ron and Hermione would take 7 days, and Luna and Hermione would take 4 days to cut the grass. The number of days it takes for them to cut it when all 3 of them work together can be written as $\frac{m}{n}$, where m and n are relatively prime integers. Find $m+n$

Solution. Notice that Luna, Ron, and Hermione are cutting at constant rates. This being the case, we can make a system of equations to represent the situation. Let L be the rate at which Luna cuts, in percentage of total field per day. Similarly, let R and H be the rates at which Ron and Hermione cut, in percentage of total field per day.

The equations will be:

$$5L + 5R = 100$$
$$7R + 7H = 100$$
$$4L + 4H = 100$$

Solving these three systems gives that

$$H = 135/14$$
L = 215 / 14
R = 65 / 14

We are asked to find the time it takes all three of them to cut the field, which is $100/(L + H + R)$
This is $100/(415/14) = 280/83$
Our answer is 363. □

Problem 17. There was a large debate concerning the order of the obstacles guarding the Sorcerer's Stone in Harry's first year. There were seven different obstacles. There were three rules, however:

1. Hagrid's obstacle (one of the obstacles) has to be placed directly before Professor Flitwick's obstacle.
2. Snape's obstacle (another one of the obstacles) has to be placed in the 4th or 5th position.
3. Professor Mcgonnagal's obstacle cannot be adjacent to Professor Quirrel's obstacle.

How many possible orders were there for the obstacles?

Solution. Let us label the 7 obstacles as A, B, H, F, S, M, and Q. H will be Hagrid's obstacle, F for Flitwick, S for Snape, M for Mcgonnagal, and Q for Quirrel.

We will begin by considering rule 2. We have 2 cases, S is either in position 4 or in position 5. Lets begin by putting S in position 5. Next, from rule 1 we know that H and F have to be next to each other in that particular order. This can be done in either locations 1 and 2, 2 and 3, 3 and 4, or 6 and 7. When H and F are placed in 1,2 or 3,4 there are 8 ways to place M and Q in the remaining 4 slots, there are 10 ways when H and F are placed in 2 and 3 and there are 6 ways when H and F are placed in positions 6 and 7. Then, there are 2 ways to place the remaining 2 obstacles. Thus we get $(8 * 2 + 6 + 10) * 2 = 64$.

Next we will place S is position 4. Then we can put H and F in 1 and 2, 2 and 3, 5 and 6, or 6 and 7. In all of these cases, there are 8 ways to place to place M and Q in all 4 of these cases. thus we get, $8 * 4 * 2 = 64$.

Lastly, $64 + 64 = 128$.

□

Problem 18. A composite number n is known to be super magical if the sum of its digits is three times its largest prime factor. What is the least three-digit super magical number?

Solution. We know that the largest sum of three digits of any three-digit number is 27. This means that the largest prime factor that we are looking for is 7. Thus, we now have to do case-work for 4 digits: 2, 3, 5, and 7.

Case 1: If we try to solve for the smallest number which has the largest prime factor equal to 2 and the sum of its digits is three times 2, we realize that the sum of the digits must be equal to 6. If the sum of a number's digits is equal to 6, it is given that 3 will be a factor, and thus, 2 cannot be the largest factor.

Case 2: Now, we want to find the smallest three-digit number which has 3 as its largest factor and the sum of its digits equal to 9. We should quickly realize that 108 has a prime factorization of $2^2 \times 3^3$. We know that there is a no three-digit number that is less than 108 which has the sum of its digits equal to 15 or 21, meaning Cases 3 and 4 can be ruled out. Thus, the answer is **108**.

□

Problem 19. Gilbert the Gecko and Octavius the Owl live on (1, 3) and (2, 2) of the coordinate plane, respectively. Their house is part of a square neighborhood with vertices (0,0), (4, 0), (4, 4), and (0, 4). Everyday, each of them travel either up, down, left, or right randomly and with equal probability by 1 unit. The probability that, after at most 3 moves, only 1 of the two leave their neighborhood (goes outside the square), while the other does not can be written as $\frac{m}{n}$, where m and n are relatively prime integers. Find $10m + n$. Note: Once someone leaves the neighborhood, they do not move anymore. Being on the border is also leaving the neighborhood.

Solution. We want to compute the probability that Gilbert gets out and Octavius doesn't, plus the probability that Gilbert does not get out but Octaviuus does. From problem 7 we know that the probability that Gilbert gets out is $\frac{3}{4}$. The probability that Gilbert gets out in 1 step is 0. After 1 step he will have a $\frac{1}{4}$ chance of getting out no matter where he moves for step 1. For step 3, if he moves to a corner spot (has $\frac{1}{2}$ chance of occurring), he has a $\frac{1}{2}$ chance of getting out. Thus, Octavius has a $\frac{1}{4} + \frac{1}{2} * \frac{1}{2} = \frac{1}{2}$. Lastly, $\frac{3}{4} * \frac{1}{2} + \frac{1}{4} * \frac{1}{2} = \frac{1}{2}$. Thus $10m + n = 12$. □

Problem 20. Square $TONK$ and triangle RON are located in planes that are perpendicular to each other. Given that $RO = 6, RN = 8$, and $ON = 10$, the length of RK can be written as $m\sqrt{n}$. Find m+n.

Solution. Since Square TONK is perpendicular to Triangle RON, we know than RN is perpendicular to NK. If we look at the diagram formed by the above constraints, we see that the value of RK can be found by using the Pythagorean theorem to find the hypotenuse of a triangle with side lengths, 8 and 10. If we solve for the $\sqrt{(10^2 + 8^2}$, we get $2\sqrt{41}$ and if we solve for the sum, we get **43**. □

Problem 21. Find the value of $\frac{x}{y}$ given the two following equations:

1. $(\log_{16} x) + (\log_8 y^3) = 6$
2. $(\log_{16} y) + (\log_8 x^3) = 9$

Solution. To solve for the first equation: $(\log_{16} x) + (\log_8 y^3) = 6$, we can start by applying the change of base formula to get $\frac{\log_{10} x}{\log_{10} 16} + \frac{\log_{10}(y^3)}{\log_{10} 8} = 6$. This equals $\frac{1}{4}\ \frac{\log_{10} x}{\log_{10} 2} + \frac{\log_{10}(y)}{\log_{10} 2} = 6$. From this, if we multiply by $4 \log_{10} 2$, we get $\log_{10} x + 4 \log_{10} y = 24 \log_{10} 2$ or $\log_{10}(xy^4)$=$24 \log_{10} 2$. From this, we can deduce that xy^4=2^{24}.

To solve for the second equation: $(\log_{16} y) + (\log_8 x^3) = 9$, we can apply the change of base formula again to get $\frac{\log_{10} y}{4 \log_{10} 2} + \frac{\log_{10}(x)}{\log_{10} 2} = 9$. If we multiply both sides of this equation by $4 \log_{10} 2$, and we get $\log_{10} y + 4 \log_{10} x = 36 \log_{10} 2$. From this, we get yx^4=2^{36}.

Now we can divide the yx^4 by xy^4 to get $(\frac{x}{y})^3$=$\frac{2^{36}}{2^{24}}$= 2^{12}. Thus, $\frac{x}{y}$= $2^{12/3}$ which is 2^4 =**16**.

□

Problem 22. Ron has three galleons in his pocket. Two of the galleons are fair, and one galleon is double-headed. He picks a galleon at random and flips it 6 times. It comes up heads each time. The probability that he picked the galleon that was double headed can be written as $\frac{m}{n}$, where m and n are relatively prime integers. Find $m + n$.

Solution. If we set H as the probability for getting 6 heads, F as the probability for choosing a fair galleon, and U as the probability for choosing an unfair galleon. $H|F$ would be probability of getting 6 heads given that it is a fair galleon and $H|U$ as the probability of getting 6 heads given that it is an unfair galleon. To find $P(U|H)$, which is what the question is looking for, we plug in the values into the equation

$\frac{P(H|U)}{P(U)\times P(H|U)+P(F)\times P(H|F)}$. This equals $\frac{1}{(1/3)\times(1)+(2/3)\times(1/6)^6}$. If we solve for this, we find m is equal to 32 and n is equal to 33. Therefore, the sum is equal to **65**. □

Problem 23. $f(x) = \frac{4^x}{25^x} + \frac{5^x}{2^x}$. The value of $f(\frac{1}{1-\log_{10} 4})$ can be written as $\frac{m}{n}$, where m and n are relatively prime integers. Find $m+n$.

Solution. We begin by rewriting $1 - \log_{10} 4$. This is equivalent to $\log_{10} 10 - \log_{10} 4$, which is equivalent to $\log_{10} \frac{5}{2}$. The original function is asking for a function of the reciprocal of the previous logarithm. $\frac{1}{\log_{10} \frac{5}{2}}$ is equal to $\log_{\frac{5}{2}} 10$. According to logarithm properties, if you raise the base of a logarithm to the logarithm itself, it is equal to the operand of the logarithm. Therefore, $\log_{\frac{5}{2}} 10$ raised to the power of $\frac{4^x}{25^x}$, is equal to $\frac{1}{100}$, as $\frac{4}{25}$ is equal to $\frac{5}{2}^{-2}$. Finally, the second part of the original equation is equal to 10, due to the properties of logarithms. Therefore, our answer is $10 + \frac{1}{100}$, which is equal to $\frac{1001}{100}$, and so $m + n = 1101$. □

Problem 24. Point P is outside a circle of unknown radius. A tangent is drawn from point P which intersects the circle at point A. A second line is drawn from point P which intersects the circle in two points, B and C, respectively. Points A and C form the diameter of the circle. If $PA = 2$ and $PB = \sqrt{2}$, the area of the circle outside the shaded region (triangle PAC is shaded) can be written as $\frac{a\pi}{b} - \frac{c}{d}$, where a and b are relatively prime integers and c and d are relatively prime integers. Find $a + b + c + d$.

Solution. To draw this diagram, notice that angle PAC is 90 degrees. Then, by power of a point, we can write $2^2 = \sqrt{2} * x$, where x is the length of PC. We get that PC = $2\sqrt{2}$.
We see that the length of PC is twice that of PB, meaning point B must bisect PC. After this realization, drawing the diagram is trivial. Also, from the fact that angle PAC is 90 degrees, we can solve for the diameter of the circle with PA as 1 side and PC as the hypotenuse. $(2\sqrt{2})^2 = 2^2 + d^2$, $d = 2$, thus the radius is equal to 1. Finally, we can solve for the area of the circle outside the shaded triangle. Drawing a line perpendicular from the center of the circle to point B, the overlapping area is 1 quarter of the circle plus a 1 by 1 right triangle.
Our area is thus $\pi - (1/4)\pi - (1/2)$, $(3/4)\pi - (1/2)$.
Our final answer is $3 + 4 + 1 + 2 = 10$ □