Problem 1. If each of the players played this game rationally, what would the end result be for the following diagram?

Solution. If player 1 chooses path A, the possible end result for him or her consists of 1, 2, 3, and 4. If player 1 chooses path B, the possible end result for him or her consists of 5, 6, 7, and 8. The numbers for path B are all greater than those of path A. This means that player 1 will chose path B.

Player 2 can choose between paths E and F now. The possible end results for player 2 consists of 3 and 2 for path E and 2 and 1 for path F. 3 and 2 are both greater than or equal to 2 and 1. This means that player 2 will choose path E.

Player 3 now has the option to choose between path K and L. Path K has an end result of 6 for player 3 and path L has an end result of 7 for player 3. 7 is greater than 6 so player 3 will choose path L.

Using all of these paths, the end result is (6, 2, 7) or path BEL. ☐

Problem 2. What strategy would result in the largest combined profits?

Solution. When choosing high price for both firms, the combined profit is $40. This is greater than $30 and $20 from the previous combinations. ☐

Problem 3. What is the Nash equilibrium in the scenario above?

Solution. There is no Nash equilibrium in the above payoff matrix. Let's look at each box. When both firms choose the low price, they both have $10 profits. If firm A changes to high price, the firm will make $25, which is more money. This means that low price, low price is not a Nash equilibrium.

If both firms choose high price, they both have $20 profits. If firm A switches to low price, the firm will have $25 profit, which is greater than $20. This means that high price, high price cannot be a Nash equilibrium.

If firm A chooses high price and firm B chooses low price, the firms profit $25 and $5 respectively. If firm B changes to high price, it will make $20 profits, which is greater than $5. This means that high price, low price is not a Nash equilibrium.

The last option is when firm A chooses low price and firm B chooses high price. This would lead to the firms making $25 and $5 profits respectively. If firm B were to change to low price, the firm would make $10 profits, which is greater than $5. Thus, this cannot be a Nash equilibrium. This means that there are zero Nash equilibrium in this payoff matrix. ☐

Problem 4. There has been a change in the payoff matrix with Firm A and Firm B. Using this new matrix, what strategy would result in the least combined profits?

Solution. When choosing low price for both firms, the combined profit is $20. This is less than $30 and $40 from the previous combinations. ☐

Problem 5. What is the difference in the profits in the Edison Cafe and Woodbridge Coffee if both places chose their prices reasonably?

Solution. Both places will choose low price. If Edison Cafe chose to switch to high price, they would go from \$90 to \$80 in profit and thus not want to change. Additionally, if Woodbridge Coffee Shop were to switch from low price to high price, they would go from \$90 to \$70 in profit and thus not want to change. This means that both places would profit \$90 and thus the difference between the two firms is \$0. □

Problem 6. How many Nash equilibria exist in this game in the matrix above? Player 1 picks the row, while Player 2 picks the column. The first coordinate corresponds to Player 1 and the second, Player 2.

Solution. If the end result is 0,0 neither player would want to change. If player 1 changed they would go from \$0 to \$0 which is not an increase. If player 2 changed, they would go from \$0 to \$0. This means that 0,0 is a Nash equilibrium.

If the end result is 10,0 neither player would want to change. If player 1 changed they would go from \$10 to \$2 which means that player 1 would not want to switch. If player 2 changed they would go from \$0 to \$0, which is not an increase. This means that 10,0 is a Nash equilibrium.

If the end result is 0,10 neither player would want to change. If player 1 changed, they would go from \$0 to \$0. If player 2 changed, they would go from \$10 to \$2. This means that 0,10 is a Nash equilibrium.

If the end result is 2,2 both players would want to change. If player 1 were to change, they would go from \$2 to \$10. If player 2 were to change they would go from \$2 to \$10. This means that 2,2 is not a Nash equilibrium.

There are a total of 3 Nash equilibria. □

Problem 7. For what values of X does a Nash equilibrium exist?

Solution. When $x \geq 3$, there is an equilibrium at row 2, column 2. When X is 3 or more player 2 would not want to switch to column 1 as $3 = 3$ and Player 1 would not switch because $1 = 1$. □

Problem 8. Mike and Jeff lent awards that they won to their friend Parth just for a weekend. Mike and Jeff came back to find that Parth lost both awards, one for each person. These awards are identical. Parth wants to know how much these awards are worth so that he can pay them back. He separates Mike and Jeff, asking them to say the price of their award from \$5 to \$20. If they say the same number, that is the true price of the awards. If one writes a smaller number, it will be taken as the true amount of money. The person who said this price will get an extra \$2 for their honesty and the other will get minus \$2. What is the Nash Equilibrium in this scenario?

Solution. Imagine that Mike and Jeff chose to put down \$20. Mike realizes that he can get more money if he puts down \$19. This is because with the \$2 bonus, he will receive \$21. Jeff realizes that Mike might do that and thus believes that putting down \$18 is the best option as it will give him \$20 rather than \$19. This process continues until both Mike and Jeff put down \$5. This means that \$5 is the Nash equilibrium. This problem is very similar to the travelers dilemma. If you would like to learn more about it, visit the Wikipedia for the travelers dilemma. □

Problem 9. What is the dominant strategy for each player?

Solution. If player 2 chose to stay, it would be best for player 1 to swerve. If player 2 chose to swerve, it would be best for player 1 to stay. This means that player 1 has no dominant strategy.

If player 1 chose to stay, it would be best for player 2 to swerve. If player 1 chose to swerve, it would be best for player 2 to stay. This means that player 2 has no dominant strategy. □

Problem 10. Player 2 has developed a strategy of staying x% of the time. Player 1 knows this strategy, but remains indifferent in their choice. What is x?

Solution. The fact that player 1 remains indifferent to their strategy means that the expected value of his result from him staying is equal to the expected value of his result from swerving. The expected values of payoff for player 1 are below.

E(stay) $= x(-10) + (1-x)3$
E(swerve) $= x(-3) + (1-x)0$
$x(-10) + (1-x)3 = x(-3) + (1-x)0$
$x = .30, 30\%$ □

Problem 11. 5 Academy students have 50 awards amongst themselves. The oldest student will propose how to split the awards. Then, the students vote (including the proposer). If $\geq 50\%$ vote yes, the split is passed. Otherwise, the proposer is asked to leave, and the process is repeated with the students that remain. (A proposer would rather get 0 awards than have to leave).

All the academy students are intelligent, and each student will vote yes only if they get more awards than they would if they voted no. What will the eldest student propose to get the most amount of awards, while also not getting kicked off?

Solution. Let's label the students A,B,C,D,E from oldest to youngest. We already know from the example, that if C, D, and E are remaining, then the split will be 49:0:1. Now, assume B, C, D, and E are remaining. B just needs 1 vote (other than himself), and can get D's vote by giving him 1 award, as D would not get any awards if B leaves. Thus, B's proposed split would be 49:0:1:0. Finally, A needs 2 votes other than himself, and can get C and E's votes by giving them 1 award each. This would be a split of 48:0:1:0:1. □

Problem 12. Repeat of problem 11. However, this time, a vote needs an absolute majority to be passed (i.e., it must be more than 50%). What will the eldest student propose this time?

Solution. We work backwards again. If E is remaining, he would get all the awards. If D and E are then remaining, D can never get E's vote, as he can't give him all the awards. If C, D, and E are remaining, then C already has D's vote, as D doesn't want to get kicked off, and thus C can give all the awards to himself. C's split would thus be 50:0:0. If B is remaining, then B can get D and E's votes by giving them 1 award each, for a split of 48:0:1:1. Finally, A can get C and E's vote with a 47:0:1:0:2 split. □

Problem 13. Repeat of problem 11. However, this time, there is just 1 award. And now, a new student has joined (6 total). What will the eldest student propose now (give all possible solutions).

Solution. Let's reduce it again. If E and F are remaining, E would get the award. If D, E, and F are remaining, D would give F the award. If C, D, E, and F are remaining, C would give either D or E the award. Then, if B through F are remaining, B can never get 3 votes. He would only get his own vote and whoever he gives the award to. Thus, if A through F is remaining, A already has B's vote, as he doesn't want to get kicked off. A can get the remaining vote by giving it to one of C-F. □

Problem 14. In this game, a player can move 1, 2, or 3 matchsticks on their turn. The pile is shown as follows (1 pile with 7 matchsticks). Does player A have a winning strategy? If so, describe it. If not, what is the winning strategy for B?

Solution. Yes. If < 4 matchsticks remain, then the player that moves next wins. If exactly 4 matchsticks remain, then the player that moves next loses. Whatever amount of matchsticks he removes, the other player will remove the rest. Thus, A wants to bring the matchsticks down to 4, which they can do by removing 3 matchsticks. □

Problem 15. Extension of problem 14. Name all piles of size 1,237,233 to 1,237,266 inclusive that have a winning strategy for player B, but not player A.

Solution. The plan from problem 14 works for all multiples of 4. Thus, if A can bring the pile down to a multiple of 4, they would win. The only time they can't do that is if the pile is a multiple of 4 to begin with. Thus, the answer is all multiples of 4 between 1,237,233 to 1,237,266. □

Problem 16. Two people are playing a game with four matchsticks. The rules are that each player can only take one or two matchsticks from a pile during any given turn. Player 2 claims that in the following arrangement with all four matchsticks in one pile, Player 1 will always win. How should Player 2 change the arrangements of matchsticks so that he or she will always win? (They can change the number of piles and number of matchsticks in each pile but not the total number of matchsticks.)

Solution. An easy solution is 4 piles of 1 matchstick. It is clear that each player would be removing 1 pile per turn, and thus after player B makes his 2nd turn, all piles would be gone. Another solution is also 2 piles of 2 matchsticks. This is because B can simply match what Player A does in the opposite pile. □

Problem 17. For this problem, the players have the ability: fast hands. With fast hands, a player can quickly swipe away as many matchsticks as they desire (but it must be at least one) from a singular pile. Take the pile below. Is there a winning strategy for A? If so, what is it. If not, what is the winning strategy for B?

Solution. The key here is that if the 2 piles are equal, whoever moves 2nd wins. This is because whoever moves 2nd can simply match the 1st player's move in the opposite pile, and thus guarantee that they will finish last. This means that player A must get both piles to be equal, which they can do by removing 1 matchstick from the left pile. □

Problem 18. Extension of problem 17. Name all ordered pairs (pile 1, pile2) where pile 1 and pile 2 are of size 1,237,233 to 1,237,240 inclusive that have a winning strategy for player B, but not player A.

Solution. We use the same idea from the previous problem. If two piles are of equal size, the second player wins. Thus, the answer is all ordered pairs where the two values are the same. This is (1,237,233 , 1,237,233), (1,237,234 , 1,237,234), (1,237,235 , 1,237,235), (1,237,236 , 1,237,236), (1,237,237 , 1,237,237), (1,237,238 , 1,237,238), (1,237,239 , 1,237,239), (1,237,240 , 1,237,240). □

Problem 19. Alice and Bob play the game again, but the number line goes from 0 to 50. Alice goes first. Is there an optimal strategy for her, and if so, what is it? If not, what is Bob's optimal strategy?

Solution. Yes, Alice has an optimal strategy. This is similar to the matchstick game, in that if there are 2 "piles" of equal size, the player that moves second wins. In this case, a pile is a division. If Alice divides the range in 2 parts, whatever Bob does in one part, Alice does in the other part. To split the range into 2 parts, A would need to pick 25. □

Problem 20. John and Bob are playing a game on an integer number line from 1 to 64, inclusive. John comes up with a number in his head and Bob must guess the number. The way that the guessing works is that Bob guesses an integer between 1 and 64 and John tells him whether the number Bob picked is higher or lower than John's number. If the number is the same, then John tells Bob that he has guessed it correctly. An example of the game is as follows: John decides that his number is 12.
Bob guesses 24 and John says lower.
Then Bob guesses 5 and John says higher.
Then Bob guesses 12 and John says that he is correct.
When John and Bob are playing this game, John realizes that no matter what he always loses and tells Bob that he needs to decide on a cap for the number of times he can guess before he loses. What is the minimum number Bob should say to ensure that he always wins?

Solution. The optimal strategy for Bob is to always guess the middle number. In this case, that would be either 32 or 33, and John says what half his number's in. By doing so, he will always be dividing the range in half. Eventually, the range will no longer be able to be divided in half, and Bob will land on the number. So this question is the same as asking how many times can we divide 64 in half? Because 64 is the same as 2^6, the answer is 6. □