MCAMC: Team Round

This section of the competition is to be completed by **your team** within **1 hour**. This section consists of **20 questions**. No calculators, notes, compasses, smartphones, smartwatches, or any other aids are allowed. All answers must be submitted on the Google Form to receive credit. Answers must be exact (do not approximate π) and in simplest form, with all fractions expressed as improper fractions. Examples of unacceptable answers include: $\frac{4}{6}$, $1\frac{1}{3}$, $3+2$. Examples of acceptable answers include: $\frac{2}{3}$, $\frac{4}{3}$, 5. There is no need to include units for any answer, and the units are always assumed to be the units in the question. Either exact decimal answers or improper fractions will be accepted (i.e. 0.25 and $\frac{1}{4}$ are both acceptable). Some questions will require a brief explanation. Additionally, questions may have no answer. If so, the correct response is "None".

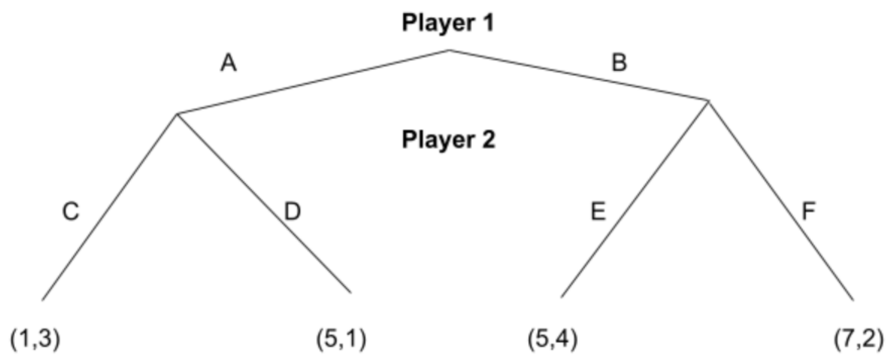
1 What is Game Theory?

Game theory is the process of modeling the strategic interaction between two or more players in a situation containing set rules and outcomes.

The main difference between game theory and probability is that in game theory, there are players who have the ability to make rational decisions in order to maximize their chances of "winning." While game theory can be used to describe games, it can also be used in economics.

With that said, let's get started.

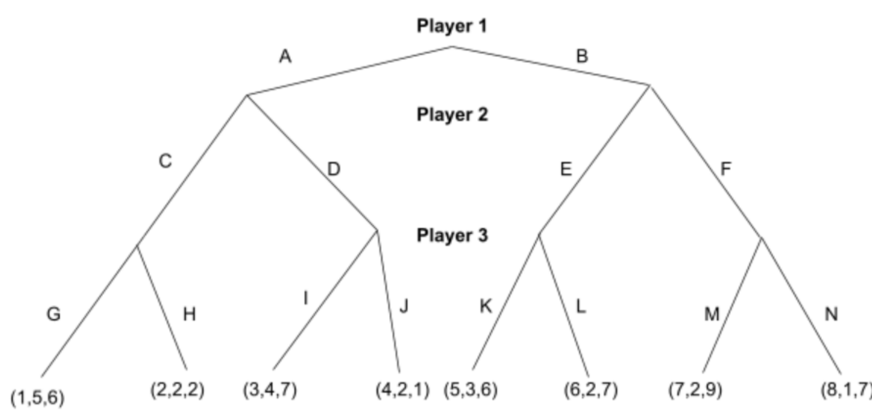
Below is a very simple example of game theory using a tree diagram.



This diagram represents a series of two moves. The first one is made by player 1. They will select to go on path A or B. Depending on the path that player 1 chooses, player 2 will be limited to only two possible moves (C,D if player 1 chooses A and E,F is player 1 chooses B). The goal of each player is to maximize their number at the end. The first coordinate is the number for player 1 and the second for player 2. For example, if player one chose path A and player chose path C, player 1 would end with a 1 and player 2 would end with a 3.

Now what should player 1 do to maximize his or her number? The possibilities for player 1 after selecting path A are 1 and 5. The possibilities after selecting path B are 5 and 7. The numbers from path B are both greater than or equal to those of path A. Thus, path B is the best opinion. From there, in order for player 2 to maximize their number, they will select path E as 4 is greater than 2. If both players were to play this "game" rationally, the end result would be (5,4).

Problem 1. If each of the players played this game rationally, what would the end result be for the following diagram?



2 Payoff Matrices

Payoff matrices are an easier way to represent two player's choices. Here, the players are Alice and Bob, each of which can choose Action 1 or 2. Their respective points depend on their own action as well as the competing player's choice. Alice's number corresponds to the first coordinate while Bob's is the second.

		Bob	
		Action 1	**Action 2**
Alice	**Action 2**	**(3,2)**	**(4,4)**
	Action 1	**(1,1)**	**(2,3)**

What strategy would result in the largest combined profits?

Clearly, both players picking action 2 are the largest, with a total of 8 points.

Definition. Nash equilibrium is the point where neither player has an advantage to move. For instance, at the top right box, neither player would want to change their action. If either player changes, they would go from 4 points to 2 points, which is unfavorable, thus they are in Nash equilibrium. Note that the goal of each player is to maximize their own score, not decrease the score of the other player.

On the other hand, at the top left box, Bob has an incentive to switch to action 2, to go from 2 points to 4 points. This is significant because the Nash equilibrium represents the constant point, where nothing is changing. Neither player switches their choice, and thus the system is in a state of steadiness.

Payoff Matrix

		Firm B	
		Low Price	High Price
Firm A	Low Price	10 10	5 25
	High Price	5 25	20 20

Here, the "players" are Firm A and Firm B, each of which can choose to price high or low. Their respective profits depend on their own price as well as the competing firm's price. The payoff of Firm A is in the bottom left corner of each box while the payoff of Firm B is in the top right corner of each box.

Problem 2. What strategy would result in the largest combined profits?

Problem 3. What is the Nash equilibrium in the scenario above?

Payoff Matrix

Firm A		Firm B: Low Price	Firm B: High Price
	Low Price	10 10	5 25
	High Price	25 5	20 20

Problem 4. There has been a change in the payoff matrix with Firm A and Firm B. Using this new matrix, what strategy would result in the least combined profits?

Use the following payoff matrix to answer the following questions. The Edison Cafe and Woodbridge Coffee Shops are the only places in Middlesex County that sell coffee. Each place can either sell at a high or low price. Depending on what they chose, the profits are shown below. The first coordinate corresponds to the Edison Cafe, while the second corresponds to the Woodbridge Coffee Shop. Both places are aware of this payoff matrix.

Woodbridge Coffee Shop

Edison Cafe	High Price	Low Price
High Price	\$100, \$110	\$80, \$140
Low Price	\$130, \$70	\$90, \$90

Problem 5. What is the difference in the profits in the Edison Cafe and Woodbridge Coffee if both places chose their prices reasonably?

Player 2

Player 1		
	0,0	10,0
	0,10	2,2

Problem 6. How many Nash equilibria exist in this game in the matrix above? Player 1 picks the row, while Player 2 picks the column. The first coordinate corresponds to Player 1 and the second, Player 2.

		Player 2	
Player 1		1, 2	0, 1
		3, 0	X, 1

Problem 7. For what values of X does a Nash equilibrium exist?

Problem 8. Mike and Jeff lent awards that they won to their friend Parth just for a weekend. Mike and Jeff came back to find that Parth lost both awards, one for each person. These awards are identical. Parth wants to know how much these awards are worth so that he can pay them back. He separates Mike and Jeff, asking them to say the price of their award from \$5 to \$20. If they say the same number, that is the true price of the awards. If one writes a smaller number, it will be taken as the true amount of money. The person who said this price will get an extra \$2 for their honesty and the other will get minus \$2. What is the Nash Equilibrium in this scenario?

3 Dominant Strategy

Definition. Dominant Strategy is the strategy a player should choose regardless of the other player's strategy. A dominant strategy occurs when no matter what the other player chooses, a specific option is consistently the better one.

There is this car game that you should never play called Chicken. The basics of the game is that two people in cars drive towards each other. If one person swerves while the other stays on their path, the one who swerves is considered the chicken and thus loses. The one who stays is considered the victor. However, if both parties continue going straight, they both lose for obvious reasons. Below is a payoff matrix for this situation.

Player 2

Player 1		Stay	Swerve
	Stay	(-10, -10)	(3,-3)
	Swerve	(-3,3)	(0,0)

Problem 9. What is the dominant strategy for each player?

Problem 10. Player 2 has developed a strategy of staying $x\%$ of the time. Player 1 knows this strategy, but remains indifferent in their choice. What is x?

4 Distributions

Distribution problems are a section of game theory in which we analyze how players deal with allocations of resources for not just themselves, but for others as well.

Example. Imagine there are 3 pirates, and 10 gold coins. The eldest pirate proposes a way to split them, and all 3 pirates vote. If $\geq 50\%$ vote yes, the split is passed, otherwise the pirate who proposed it has to walk the plank. The pirates will vote yes only if they get more coins than by voting no. After a pirate walks the plank, the next oldest pirate will propose a way to split them. Every pirate wants to maximize the number of coins they receive. What will the eldest pirate propose?

Solution. 9:0:1. (Oldest pirate gets 9 coins, and the youngest gets one). Why? Obviously, the eldest pirate will vote yes. Thus, all he needs to do is convince one of the other two to vote for him. We must remember that pirates will vote yes if they get more coins than by voting no. So, we must look at what happens if they vote no. If the vote fails, then the eldest pirate walks the plank, and 2 pirates remain. Then, Pirate 2 will just offer a 10:0 split in the coins, and even when Pirate 3 votes no, Pirate 2 already has at least 50% of the vote (because he has his own vote). Thus, to convince pirate 3 to vote for him, pirate 1 just has to offer him 1 coin, as that is more than he would have gotten otherwise. □

Problem 11. 5 Academy students have 50 awards amongst themselves. The oldest student will propose how to split the awards. Then, the students vote (including the proposer). If $\geq 50\%$ vote yes, the split is passed. Otherwise, the proposer is asked to leave, and the process is repeated with the students that remain. (A proposer would rather get 0 awards than have to leave).

All the academy students are intelligent, and each student will vote yes only if they get more awards than they would if they voted no. What will the eldest student propose to get the most amount of awards, while also not getting kicked off?

Problem 12 Repeat of problem 11. However, this time, a vote needs an absolute majority to be passed (i.e., it must be more than 50%). What will the eldest student propose this time?

Problem 13. Repeat of problem 11. However, this time, there is just 1 award. And now, a new student has joined (6 total). What will the eldest student propose now (give all possible solutions).

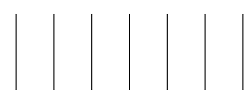
5 Matchsticks

Here, we introduce new games involving matchsticks, in which the last person to remove a matchstick wins. Let's start with an easy example.

Example. Here, 2 piles of matchsticks are represented (a line represents a matchstick and a large space represents the start of a new pile). When it is a player's turn, they may remove one or two matchsticks from one pile. Player A goes first. Do they have a winning strategy?

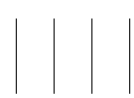
Solution. Yes. The trick is to remove one of the matchsticks from group 1. Then, player B must remove the remaining matchstick from group 1 or group 2, and player A can win by removing the last matchstick. Notice any other move would have lost for A. If player A removes the matchstick from pile 2, player B removes both matchsticks from pile 1. If player A removes both matchsticks from pile 1, player B removes the matchstick from pile 2. □

Problem 14. In this game, a player can move 1, 2, or 3 matchsticks on their turn. The pile is shown as follows (1 pile with 7 matchsticks). Does player A have a winning strategy? If so, describe it. If not, what is the winning strategy for B?



Problem 15. Extension of problem 14. Name all piles of size 1,237,233 to 1,237,266 inclusive that have a winning strategy for player B, but not player A.

Problem 16. Two people are playing a game with four matchsticks. The rules are that each player can only take one or two matchsticks from a pile during any given turn. Player 2 claims that in the following arrangement with all four matchsticks in one pile, Player 1 will always win. How should Player 2 change the arrangements of matchsticks so that he or she will always win? (They can change the number of piles and number of matchsticks in each pile but not the total number of matchsticks.)



Problem 17. For this problem, the players have the ability: fast hands. With fast hands, a player can quickly swipe away as many matchsticks as they desire (but it must be at least one) from a singular pile. Take the pile below. Is there a winning strategy for A? If so, what is it. If not, what is the winning strategy for B?

Problem 18. Extension of problem 17. Name all ordered pairs (pile 1, pile2) where pile 1 and pile 2 are of size 1,237,233 to 1,237,240 inclusive that have a winning strategy for player B, but not player A.

6 Number Lines

For this final section, we will introduce a few games involving the number line.

Example. There is a number line from 0 to 2, inclusive. Alice and Bob are playing a game where they take turns picking a number. The number must be greater than 1 unit away from all previous numbers picked. Let's go through an example game.

Alice picks the number 0.
Bob picks the number 1.5.
All points remaining are ≤ 1 unit away from either 0 or 1.
Thus, Bob wins.

Problem 19. Alice and Bob play the game again, but the number line goes from 0 to 50. Alice goes first. Is there an optimal strategy for her, and if so, what is it? If not, what is Bob's optimal strategy?

Now, Alice's friend John and Bob was to play a higher or lower game on this number line.

Problem 20. John and Bob are playing a game on an integer number line from 1 to 64, inclusive. John comes up with a number in his head and Bob must guess the number. The way that the guessing works is that Bob guesses an integer between 1 and 64 and John tells him whether the number Bob picked is higher or lower than John's number. If the number is the same, then John tells Bob that he has guessed it correctly. An example of the game is as follows: John decides that his number is 12.
Bob guesses 24 and John says lower.
Then Bob guesses 5 and John says higher.
Then Bob guesses 12 and John says that he is correct.
When John and Bob are playing this game, John realizes that no matter what he always loses and tells Bob that he needs to decide on a cap for the number of times he can guess before he loses. What is the minimum number Bob should say to ensure that he always wins?